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Rational Ermakov systems of Fuchsian type

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Abstract. We study a subclass of a class of systems of coupled nonlinear oscillator that have well-behaved singularities which can be studied using an exact linearization. This allows us to classify such systems and to isolate those with special properties.

1. Introduction

Ermakov systems are fourth-order, nonlinear, ordinary differential systems in two dependent variables, of the form

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\omega^2(t)x&=x^{-3}f\left(\frac{x}{y}\right)\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\omega^2(t)y&=y^{-3}g\left(\frac{x}{y}\right).\end{aligned} \tag{1}$$

A subclass of such systems was first introduced in [1] and the full class in [2]. In both instances the chief interest lay in the existence of an invariant, now called the *Lewis-Ray-Reid invariant*, for the system (1). This invariant allows one to construct $y(t)$ from a known $x(t)$ by a single integration and hence provides an *implicit* superposition law [3] for solutions of (1).

More recently it has been shown that the system (1) can be *linearized*, at least locally [4]. The linearization is effected in two stages. Firstly, one may reduce (1) to the *autonomous* form ($\omega^2\equiv 0$) provided one can solve the linear equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\omega^2(t)x=0. \tag{2}$$

Secondly, one exploits the symmetries associated with time translation and the Lewis-Ray-Reid invariant to reduce the order of the autonomous system to a single second-order equation. It is not clear *why* this equation should be, as it turns out to be, linear.

The fact that the autonomizing and linearizing transformations are local creates problems for the inverse procedure of reconstructing the general solution to (1). However, the autonomizing procedure can be viewed as a transformation to a *surrogate* time variable living on the complex line $\mathbb{P}_1\mathbb{C}$. The inverse of this procedure then consists in patching together the solutions defined on local coordinate patches for $\mathbb{P}_1\mathbb{C}$. Further, away from singularities of the linearization one has local analytic expansions from which one may reconstruct the local solutions to (1) and obtain *explicit* local superposition laws for solutions. Finally, the singularities of the linearized equations determine the nature of the singularities of the nonlinear system. One may then hope to classify

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Ermakov systems according to their singularity structure deduced from the singularity structure of the linearization insofar as the singularities of linear equations are sufficiently well understood [5].

One might alternatively classify Ermakov systems according to the character (rational, algebraic, automorphic, etc.) of their superposition laws.

A specific example will serve to illustrate some of the above remarks. The coupled Pinney equations

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\omega^2(t)x &= \frac{1}{x^3}\left[\beta-\alpha\left(\frac{x}{y}\right)^4\right] \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\omega^2(t)y &= \frac{1}{y^3}\left[\delta-\gamma\left(\frac{y}{x}\right)^4\right] \end{aligned} \tag{3}$$

have been discussed in [6], $\omega^2(t)$ being taken to be periodic in t of period π. In this case the linearization consists of (2) and a family of linear second-order equations depending on a single parameter I, the numerical value of the Lewis–Ray–Reid invariant, entirely determined by initial conditions $x(0)$, $y(0)$, $\mathrm{d}x/\mathrm{d}t(0)$ and $\mathrm{d}y/\mathrm{d}t(0)$. For all values of I the equations of this family are Fuchsian [5]. There are six singular points, two of which are fixed at 0 and ∞, the remaining four points varying in the complex plane with the value of I. The exponents of all these points are independent of I except at the confluences of the wandering points when their exponents jump. Confluence occurs in pairs and because the wandering singular points are elementary the equation remains Fuchsian.

In [6] it is shown that the qualitative global behaviour of solutions to (3) can be deduced from that of the linearization. Specifically, stability (periodicity) of the general integral of the linearization implies stability (periodicity) of the set of solutions to (3) having the given value of I. As I changes (i.e. as the initial conditions change) one will see changes in the qualitative behaviour of solutions to (3).

Following the above discussion we define a *rational Ermakov system of Fuchsian type* to be one for which f and g are rational functions of z and whose linearization is a one-parameter family of (rational) Fuchsian equations. We shall see that the parameter I must enter the equations in a very specific manner. The weakest excursion from this class will be equations which for some isolated value of I acquire an irregular singular point.

The paper is ordered as follows. In section 2 we will briefly review the linearization procedure, of which more details are elsewhere presented [4, 6]. Slightly different variables are employed here and we emphasize the geometric aspects of the construction. A more systematic, algebraic derivation of the linearization has since been given in [7] and we demonstrate the connection between the two approaches. Section 3 motivates the discussion of systems of Fuchsian type by showing that one can invert the linearization in a neighbourhood of a regular singular point *without logarithms* to obtain a *movable* regular singular point, without logarithms, for the nonlinear Ermakov system. In section 4 we derive the form of f and g necessary and sufficient for (1) to be of Fuchsian type. In view of the results of section 4, such Ermakov systems are *free of movable essential singularities* and hence satisfy a criterion introduced by Painlevé [8]. Section 5 is devoted to the description of rational Ermakov systems of Fuchsian type which have algebraic superposition laws. These are the *rational polyhedral Ermakov systems.*

2. Linearization

The system (1) is linearized by first removing the terms $\omega^2 x$ and $\omega^2 y$ from the left-hand sides. Let x_1 and x_2 be any two linearly independent solutions to (2) having unit Wronskian,

$$x_2 \frac{dx_1}{dt} - x_1 \frac{dx_2}{dt} = 1. \tag{4}$$

Then in the barred variables defined by $x = x_2\bar{x}$, $y = x_2\bar{y}$, $s = x_1/x_2$ the system becomes

$$\frac{d^2\bar{x}}{ds^2} = \frac{1}{\bar{x}^3} f\left(\frac{\bar{x}}{\bar{y}}\right) \qquad \frac{d^2\bar{y}}{ds^2} = \frac{1}{\bar{y}^3} g\left(\frac{\bar{x}}{\bar{y}}\right). \tag{5}$$

The autonomizability of (1) depends crucially upon the fact that the right-hand sides are homogeneous of weight -3.

Now we exploit the autonomy of these equations by making new dependent variables $p = 2\bar{x}\, d\bar{x}/ds$ and $q = 2\bar{y}\, d\bar{y}/ds$ and using as independent variable $z = \bar{x}/\bar{y}$. Interestingly this gives us a *second-order* system in p, q and z

$$(p - qz^2) z \frac{dp}{dz} - p^2 = 4f(z) \qquad (p - qz^2) \frac{dq}{z\, dz} - q^2 = 4g(z). \tag{6}$$

The Lewis-Ray-Reid invariant

$$I = \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)^2 - \int^z u^{-3} f(u)\, du - \int^{1/z} u^{-3} g(u)\, du \tag{7}$$

rewritten in the new variables, becomes an algebraic relation between p, q and z:

$$p - qz^2 = zh(z; I) \tag{8}$$

where

$$h^2(z; I) = 8I + 8\int^z (u^{-3} f(u) - ug(u))\, du \tag{9}$$

and (6) must be supplemented by the equation relating z to s,

$$\bar{x}^2(s) \frac{dz}{ds} = \tfrac{1}{2}(p - qz^2)z = \tfrac{1}{2} z^2 h(z; I). \tag{10}$$

Using (7), the pair of equations (6) becomes a pair of independent Riccati equations,

$$z^2 h \frac{dp}{dz} - p^2 = 4f(z) \qquad h\frac{dq}{dz} - q^2 = 4g(z). \tag{11}$$

Finally, the linearizing transforms $p = -z^2 h \psi^{-1}\, d\psi/dz$ and $q = -h\varphi^{-1}\, d\varphi/dz$ lead to

$$\begin{aligned} &z^4 h^2 \frac{d^2\psi}{dz^2} + z^2 h \frac{d(z^2 h)}{dz}\frac{d\psi}{dz} + 4f(z)\psi = 0 \\ &h^2 \frac{d^2\varphi}{dz^2} + h\frac{dh}{dz}\frac{d\varphi}{dz} + 4g(z)\varphi = 0. \end{aligned} \tag{12}$$

It is important to note that because $h = h(z; I)$, the above is really a *one-parameter* (I) *family* of linear differential equations. Further discussion of these equations using slightly different variables is given in [4, 6].

Equations (2) and (12) constitute the linearization *up to* the general integral of (10). In fact we may solve (10) explicitly *given* the general solution to (12) so that (2) and (12) do constitute the full linearization of the Ermakov system (1). This fact was not fully appreciated in [4]. To see this, note that we may integrate the linearizing substitutions for p and q to obtain

$$\bar{x}(s)\psi(z)=\bar{y}(s)\varphi(z)=1. \tag{13}$$

Suppose now that we had effected the autonomizing procedure leading to (5) using a different linearly independent pair of solutions x_1' and x_2' to (2). The corresponding variable s' is a homographic transformation of s,

$$s'=\frac{\alpha+\beta s}{\gamma+\delta s} \tag{14}$$

for constants α, β, γ and δ with $\alpha\delta-\beta\gamma\neq 0$. In addition,

$$x(t)=x_2'(t)\bar{x}'(s') \qquad y(t)=x_2'(t)\bar{y}'(s') \tag{15}$$

and

$$\bar{x}'\psi'(z)=\bar{y}'\varphi'(z)=1 \tag{16}$$

where ψ' and φ' also satisfy (12). Consequently,

$$\psi'(z)\psi^{-1}(z)=\varphi'(z)\varphi^{-1}(z)=\gamma+\delta s. \tag{17}$$

Thus if ψ_1 and ψ_2 are a pair of linearly independent solutions to the first of equations (12) and we define the ratio of solutions $\sigma(z)=\psi_1(z)\psi_2^{-1}(z)$, then (17) gives z as an implicit (I-dependent) function of s via

$$\sigma(z)=\frac{A+Bs}{C+Ds} \tag{18}$$

for constants A, B, C and D satisfying $AD-BC\neq 0$. Rewriting (10) as

$$\frac{\mathrm{d}z}{\mathrm{d}s}=\tfrac{1}{2}z^2h(z;\, I)\psi^2(z) \tag{19}$$

and noting that ψ contains a pair of arbitrary constants, we see that the *three-parameter* family of functions (18) defines the general integral of (10) at each value of I.

Geometrically the variable s lives on the complex projective line $\mathbb{P}_1\mathbb{C}$ and different choices of basis of the solution space of (2) correspond to different choices of affine representation for $\mathbb{P}_1\mathbb{C}$ related by homographic transformation (14). As the independent variable in the linear equations (12) z also lives on the projective line so that the ratio of solutions, $\sigma(z)$, is a map from the Riemann sphere to itself. It will be in general multivalued and only locally holomorphic. σ is labelled by the projective invariant I. Equation (18) gives the variation of x/y as a function of t. $\psi(z)$ can be reconstructed from σ using the classical formulae for independent solutions

$$\psi_1=\sigma\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^{-1/2} \qquad \psi_2=\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^{-1/2} \tag{20}$$

and combined with the relation $x\psi=x_2$, which follows from (13) and the definitions of the barred variables, to obtain for each value of I the general trajectories of solutions to (1).

In [7] the linearization is obtained through a systematic search for *time-dependent* integrals of the Ermakov system using a purely algebraic approach. Because the result is expressed in a form not *obviously* compatible with that summarized above we here make clear the connection.

The quantities (using our notation)

$$I_{ij}=\left(x_i\frac{\mathrm{d}y}{\mathrm{d}t}-y\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)\psi_j+\frac{1}{y}\left(y\frac{\mathrm{d}x}{\mathrm{d}t}-x\frac{\mathrm{d}y}{\mathrm{d}t}\right)x_i\frac{\mathrm{d}\psi_j}{\mathrm{d}z} \tag{21}$$

where i, j take the values 1, 2 and the ψ_i are linearly independent solutions to the second of equations (12), are shown to be functionally independent invariants for the Ermakov system (1) up to a single functional relation

$$I_{11}I_{22}-I_{12}I_{21}=\Psi(I). \tag{22}$$

Writing the z-derivative in (21) as a t-derivative we find

$$I_{ij}=x_i^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y\psi_j}{x_i}\right). \tag{23}$$

The projective variables $s_i=x_{\bar{i}}/x_i$ where $\bar{i}=1$, 2 as $i=2$, 1, respectively, are now defined so that $\mathrm{d}/\mathrm{d}s_i=x_i^2\,\mathrm{d}/\mathrm{d}t$. Then (23) can be integrated to give

$$y(t)=(I_{ij}x_{\bar{i}}+c_{ij}x_i)\psi_j^{-1} \tag{24}$$

for constants c_{ij}. We recognize this as (13), using the definition of the barred variables. Equation (24) consists of four equal expressions from which it follows that the c_{ij} are known from the I_{ij} and one obtains

$$y(t)=\frac{I_{12}x_2+I_{22}x_1}{\varphi_2}=\frac{I_{11}x_2+I_{21}x_1}{\varphi_1}. \tag{25}$$

Then the Wronskian relation (4) between x_1 and x_2 yields (22), and (24) can be written as

$$y(t)=\frac{\Psi(I)x_1}{I_{11}\varphi_2-I_{12}\varphi_1}=\frac{\Psi(I)x_2}{I_{22}\varphi_1-I_{21}\varphi_2}. \tag{26}$$

Multiplying by z and using the fact that $\varphi=z\psi$ we obtain the corresponding expression for $x(t)$,

$$x(t)=\frac{\Psi(I)x_1}{I_{11}\psi_2-I_{12}\psi_1}=\frac{\Psi(I)x_2}{I_{22}\psi_1-I_{21}\psi_2} \tag{27}$$

in terms of a basis ψ_1, ψ_2 of solutions to the first of equations (12). Equation (27) also implies a relation of the form (18),

$$\sigma(z)=\frac{I_{21}x_1+I_{11}x_2}{I_{22}x_1+I_{12}x_2}. \tag{28}$$

So the invariants I and I_{ij} are explicitly seen to be the constants of integration arising through linearization and, as they must be, the four independent constants of integration present in the general solution. Equations (26), (27) and (21) give a convenient representation (locally) of the general solution to (1).

It is apt to regard these equations as nonlinear superposition formulae for the general solution of (1) in terms of the general solution of the time-dependent harmonic oscillator. The superposition law will be rational, algebraic, etc., according to the character of the function $\sigma(z)$, the character of which alters, in general, with changes in the value of I.

As a closing remark to this section let us put (12) into a more evocative form by introducing a new variable τ defined, following [3], by $d\tau = y^{-2}\,dt$. Then we obtain

$$\frac{d^2\varphi}{d\tau^2} + G(\tau)\varphi = 0 \tag{29}$$

where $G(\tau) = g(z(\tau))$, $z(\tau)$ being the solution to the nonlinear oscillator

$$\frac{d^2 z}{d\tau^2} = z^{-3} f(z) - zg(z). \tag{30}$$

It is easy to choose forms for f and g which make (29) and (30) into well-known equations, for instance, Lamé's equation and the equation for the Weierstrass $\wp$-function.

3. Singularity structure

In this section we discuss the analytic inversion of the linearization near-ordinary and regular singular points of the linearized equations.

Suppose firstly we are at an ordinary point in both cases. Then $x_1(t)$, $x_2(t)$, $\psi_1(z)$ and $\psi_2(z)$ have local analytic expansions near the ordinary point (t_0, z_0). Define $s = x_1/x_2$, $s' = x_2/x_1$, $\sigma = \psi_1/\psi_2$ and $\sigma' = \psi_2/\psi_1$. Since t_0, z_0 is an ordinary point, zeros of x_1 and x_2 are simple and do not coincide in a neighbourhood of t_0; likewise for ψ_1 and ψ_2 in a neighbourhood of z_0. From (18) we have, without loss of generality,

$$\sigma(z) = s(t) \qquad \sigma'(z) = s'(t) \tag{31}$$

on the patches $s \neq \infty$ and $s' \neq \infty$ of $\mathbb{P}_1\mathbb{C}$ respectively. Hence the zeros of $\psi_1(z)$ coincide with those of $x_1(t)$ and those of $\psi_2(z)$ with those of $x_2(t)$. Hence we may invert (31) to obtain $z = f(t)$, locally analytic, with $z_0 = f(t_0)$. Then

$$x(t) = \frac{x_2(t)}{\psi_2(z)} = \frac{x_1(t)}{\psi_1(z)} \tag{32}$$

and

$$y(t) = \frac{x(t)}{z(t)} \tag{33}$$

define locally analytic functions. A more cumbersome argument to this effect is given in [4]. Note that $z = 0, \infty$ are bound not to be ordinary points of *both* equations (12) so that $z(t_0) \neq 0, \infty$ and $x(t)$, $y(t)$ do not cross the x-, y-axes for t in the relevant neighbourhood of t_0.

Suppose now that t_0 is an ordinary point but that z_0 is a regular singular point, ψ having exponents υ_1 and υ_2 but no term involving logarithms. Then there exist linearly independent solutions,

$$\psi_i(z) = (z - z_0)^{\upsilon_i}\phi_i(z) \qquad i = 1, 2 \tag{34}$$

where the ϕ_i are analytic near and non-vanishing at z_0. Put $\delta = \upsilon_1 - \upsilon_2$. Then, by (18),

$$\sigma(z) = (z - z_0)^{\delta}\Phi(z) \tag{35}$$

Φ being analytic near and non-vanishing at z_0. Assume without loss of generality that $\delta > 0$. Certainly $\delta \neq 0$ or we would have logarithms. We may invert (18), making a choice of branch, to obtain an expansion for z in terms of the variable $w = s^{1/\delta}$, $s(t_0) = 0$ for suitable x_1, x_2:

$$z = z_0 + w\tilde{\Phi}(w) \tag{36}$$

where $\tilde{\Phi}$ is analytic and non-vanishing at $w = 0$. So

$$x(t) = x_2^{\upsilon_1/\delta} x_1^{-\upsilon_2/\delta} X(w(t)) \tag{37}$$

where $x_1(t_0) = 0$. Similarly

$$y(t) = x_2^{\upsilon_1/\delta} x_1^{-\upsilon_2/\delta} Y(w(t)) \tag{38}$$

where $X/Y \to z_0$ as $t \to t_0$. Equations (37) and (38) describe the leading (movable) singularity of the solution. The monodromy of the singularity is determined by the ratio υ_1/υ_2 of the exponents at the linearized singularity

$$x(t), y(t) \approx (t - t_0)^{\rho} \tag{39}$$

where $\rho = (1 - \upsilon_1/\upsilon_2)^{-1}$. In the case of an elementary regular singular point $\upsilon_1 = \frac{1}{2}$, $\upsilon_2 = 0$, so $\rho = 0$, $\delta = \frac{1}{2}$ and $x(t)$ and $y(t)$ are analytic functions of $(x_1/x_2)^2$ not vanishing at $t = t_0$. In this case the singularity evaporates [6].

If the linear equations (12) are Fuchsian (all singular points, including the point at infinity, if singular, are regular singular points) then, in the absence of logarithms, we can carry through the above analytic inversion procedure to obtain the general solution to (1) together with a description of its singularities. The character of the singularities may or may not vary with I.

Similar arguments to those presented here will apply when t_0 is a regular singular point of (2) and when both t_0 and z_0 are regular singular points. However, since the solution of (2) is a problem independent of the form of the functions f and g which, through the autonomous form, characterize whole classes of systems (1), we regard this problem as subsidiary and the above discussion of singularities only of (12) sufficient.

In the case that the singularity at z_0 in (12) is an *irregular* one the functions $\phi_i(z)$ are single valued but no longer analytic at z_0. One would expect this situation to reflect itself in *movable essential singularities* in the general solution $x(t)$, $y(t)$ but this requires further investigation.

4. Rational Ermakov systems of Fuchsian type

We take f and g to be rational functions such that $h^2(z; I)$ is also rational and free of logarithms (i.e. $z^{-3}f(z) - zg(z)$ is free of simple poles). Define $h^2 \equiv 8H$. Then

$$H = I + \frac{P}{Q} \tag{40}$$

P and Q being coprime polynomials in the variable z. Their zeros are fixed (independent of the value of I). The polynomial $P + IQ$ has but simple poles, except at discrete values of I. If this was not the case both it and its derivative with respect to I must vanish for some $z=z(I)$, which contradicts the coprimality of P and Q. The rational functions P/Q and g determine f and we thus seek conditions upon them that, *for all values of I*, the linearization (12) is a pair of Fuchsian equations. Since φ and ψ in (12) are related by $\varphi=z\psi$ it suffices to choose the simpler of the two equations, that for φ, which in terms of H is written as

$$2H\varphi''+H'\varphi'+g\varphi=0 \tag{41}$$

d/dz now denoted by a prime.

The (semi)canonical form of a second-order ordinary differential equation,

$$\varphi''+p\varphi'+q\varphi=0 \tag{42}$$

is defined to be

$$\phi''+J\phi=0 \tag{43}$$

where

$$J=q-\tfrac{1}{2}p'-\tfrac{1}{4}p^2 \tag{44}$$

is the (relative projective) invariant of (42). This invariant transforms under a change of independent variable $z\to Z(z)$ as

$$J\to J(Z)Z'^2+\tfrac{1}{2}\{Z,z\} \tag{45}$$

where

$$\{Z,z\}=\frac{Z'''}{Z'}-\frac{3}{2}\frac{Z''^2}{Z'^2} \tag{46}$$

is the Schwarzian derivative of Z with respect to z. The invariant then transforms homogeneously under homographic (projective) transformations and only under such transformations.

The Fuchsian class of second-order equations is given by (42) with p and q of the forms

$$p=\sum_1^n\frac{p_i}{z-z_i}\qquad q=\sum_1^n\frac{q_i}{(z-z_i)^2}+\frac{R_0+R_1z+\ldots+R_{n-2}z^{n-2}}{\Pi_1^n(z-z_i)} \tag{47}$$

where $z_i\neq z_j$ for $i\neq j$. The class is characterized by the property that every singular point, including the point at infinity, if singular, is a *regular* singular point. Some or all of the p_i and q_i may take the value zero, the point being that the poles of p (q) are of order at most one (two) and finite in number.

It is easy to see using (45) and the invariant constructed from (47) that the Fuchsian class is preserved under rational transformations $z\to Z(z)$.

Our requirement is that the invariant for (41), namely

$$J=\frac{1}{2}\frac{g}{H}-\frac{1}{4}\frac{H''}{H}+\frac{3}{16}\frac{H'^2}{H^2} \tag{48}$$

be of the form of q in (47) *for all values of I*. It is easy to see that the terms in (48) which involve derivatives of the rational function H satisfy the requirement whatever the value of I. Poles will be present at the zeros of Q and $P+IQ$.

Singularities are of *three* kinds: those whose location in the z-plane does not depend on the value of I (the *fixed* singular points), those whose location changes with the value of I (the *movable* singular points), and those which arise as confluences of movable singular points for isolated values of I (we will also call these *fixed*).

Now consider a value of I for which $P+IQ$ separates into linear factors,

$$P+IQ=\prod_1^l (z-a_i) \qquad Q=\prod_1^m (z-b_i)^{\kappa_i} \tag{49}$$

where the κ_i are arbitrary, strictly positive integers. Then the remaining term in J is

$$J_0 \equiv \frac{gQ}{2(P+IQ)} \tag{50}$$

where g and Q are both I independent so that the l movable zeros of $P+IQ$ *all* give rise to poles of the first order. Put

$$gQ=\frac{2\Gamma}{\Lambda \prod_1^m (z-b_i)^2}. \tag{51}$$

Here Γ and Λ are coprime polynomials. Λ, but not necessarily Γ, is to be coprime to Q and must have zeros of order at most two. Put $\Lambda=\Lambda_0\Lambda_1^2$ where Λ_1 and Λ_2 are separable and coprime. In order that J_0 satisfy the condition of regularity at $z=\infty$ the degrees of Γ and Λ must satisfy the inequality

$$\deg\Gamma - \deg\Lambda \leqslant 2m+l-2. \tag{52}$$

J_0 is now of the form

$$J_0=\frac{\Gamma}{\Lambda_0\Lambda_1^2(P+IQ)\Pi(z-b_i)^2}. \tag{53}$$

Next we must examine the requirement that J_0 have at most second-order poles at values of I for which $P+IQ$ has zeros of order $s+1$, $s>0$. These zeros of $P+IQ$ are confluences of (simple) movable zeros. These fixed zeros are in one-to-one correspondence with those of the function

$$S \equiv \text{Numerator}\left(\frac{P'Q-Q'P}{Q^2}\right). \tag{54}$$

If Q has only simple zeros then $S=P'Q-Q'P$. (Note that zeros of P are zeros of H for $I=0$ but that zeros of Q are not zeros of H for finite values of I.) If $P+IQ$ has a zero of order $s+1$ then S has a zero of order s. Since Γ and Λ are coprime we must demand that Λ_1 and $P+IQ$ be coprime for all I. But Λ_0 *may* have factors in common with $P+IQ$ for certain I.

Regarding the zeros of S, either Λ_0 is coprime to S or one of its linear factors is repeated to some power in S. In the latter case this the factor must appear in Γ to at least the first power in order to cancel the high-order zero in the denominator. This contradicts the coprimality of Γ and Λ. Hence Λ_0 is coprime to S, and Γ must have a factor

$$\prod_1^d (z-z_i)^{s_i-1} \tag{55}$$

where the z_i, $i=1,\ldots,d$ are the zeros of S having orders s_i. One can say no more beyond this without knowing more about the forms of P and Q.

In summary: Rational Ermakov systems of Fuchsian type are characterized by polynomials P, Q, Γ and Λ which are arbitrary except insofar as they satisfy the following conditions:

(i) $(P, Q) = (\Gamma, \Lambda) = (Q, \Lambda) = 1$;

(ii) $\deg \Gamma - \deg \Lambda \leqslant \max(\deg P, \deg Q) + 2(\text{No. of zeros of } Q) - 2$;

(iii) $\Lambda = \Lambda_0 \Lambda_1^2$, Λ_0 and Λ_1 separable; $(\Lambda_1, P + IQ) = 1$, $\forall I$; $(\Lambda_0, S) = 1$, where $S =$ Numerator $(P'Q - Q'P)/Q^2$;

(iv) Order s zeros of S are order $s-1$ zeros of Γ.

(Here (A, B) denotes the highest common factor of A and B.)

A simpler class of systems is given by the choice $\Lambda = 1$:

(i)′ $(P, Q) = 1$;

(ii)′ $\deg \Gamma \leqslant \max(\deg P, \deg Q) + 2(\text{No. of zeros of } Q) - 2$;

(iv) Order s zeros of S are order $s-1$ zeros of Γ.

One way of satisfying (iv) is to choose $\Gamma = P''Q - Q''P$ but this is unnecessarily heavy handed. A better way is to demand that $P'Q - PQ'$ be separable, which in turn requires that P and Q have zeros of order at most two.

Thus, the coupled Pinney system [6] is given by the choice $f(z) = \alpha - \beta z^4$ and $g(z) = \gamma - \delta z^{-4}$ from which $H = I - z^2/2 - 1/2z^2$, with $\alpha + \delta = \beta + \gamma = 1$ by suitably scaling x and y. So $P = -1 - z^4$, $Q = 2z^2$ and $\Gamma = \gamma z^4 - \delta$. Both (i)′ and (ii)′ are easily seen to be satisfied and, since $P'Q - Q'P = 2z(1 - z^4)$ has only simple zeros, (iv) is trivially satisfied. We therefore define the class of *generalized coupled Pinney equations* defined by polynomials P, Q and Γ satisfying the following simple conditions:

(i)′ $(P, Q) = 1$;

(ii)′ $\deg \Gamma \leqslant \max(\deg P, \deg Q) + 2(\text{No. of zeros of } Q) - 2$;

(iv)′ $P'Q - PQ'$ has only simple zeros.

For this subclass only *pairwise confluence* of movable singularities is possible and the fixed singularities are these confluences together with the zeros of P and Q.

Another subclass of interest, and to which some of the systems in the next section belong, arise by taking $P = F^n$, $Q = G^n$ where F and G are coprime and $F'G - G'F$ has only simple zeros. Γ must have a factor of $nF^{n-1}F'' - n(n-1)F^{n-2}F'^2$.

Finally, in order to apply the results of the previous section we need to check that logarithms do not arise. It is sufficient for this that the exponents of a singularity do not differ by an integer. In cases where the difference *is* an integer, no *simple* general criterion seems to exist and one must check the absence of logarithms in any specific case using an algorithm [5] which is in general tedious. From (49) we see that the exponents of the movable singularities always satisfy the indicial equation

$$v(v-1) + \tfrac{3}{16} = 0 \tag{56}$$

so that the difference is $\frac{1}{2}$ and the singular point is an elementary singular point. But the exponents of the fixed singularities, in particular the confluences of movable singularities, depend upon the details of P, Q, Γ and Λ and the value of I.

5. Rational Ermakov systems of algebraic type

Within the class of rational Ermakov systems of Fuchsian type will be a subclass with the special property that their general solutions are algebraic functions of the solutions to the linear time-dependent oscillator. These will have linearizations whose general solutions are algebraic functions of z. Rational second-order Fuchsian equations having algebraic integrals were classified by Klein [9] and are associated with finite subgroups of $\mathrm{psl}(2, \mathbb{C})$ i.e. the symmetry groups of the regular three-dimensional solids. Hence we call such Ermakov systems *polyhedral Ermakov systems*. An incomplete set of examples is given in [10]. In this final section we systematically isolate such systems.

In order to have algebraic integrals for all I the invariant J given in (48) must be equivalent by a rational transformation to one of the following forms:

$$J_{\mathrm{X}} = \frac{1}{4}\left[\left(1 - \frac{1}{\upsilon_2^2}\right)\frac{1}{z^2} + \left(1 - \frac{1}{\upsilon_1^2}\right)\frac{1}{(z-1)^2} + \left(\frac{1}{\upsilon_1^2} + \frac{1}{\upsilon_2^2} - \frac{1}{\upsilon_3^2} - 1\right)\frac{1}{z(z-1)}\right] \tag{57}$$

where X = I, II, III, IV or V and the integer exponents $(\upsilon_1, \upsilon_2, \upsilon_3)$ are given by the following values:

Case I	$(N, 1, N)$, cyclic
Case II	$(2, 2, N)$, dihedral
Case III	$(2, 3, 3)$, tetrahedral
Case IV	$(2, 3, 4)$, octahedral
Case V	$(2, 3, 5)$, icosahedral.

We have taken the liberty of writing case I in the form (57), contrary to usual practice [11], at the expense of spoiling the ascending order of the υ_i in that case. Thus we seek to find a rational function $Z(z)$ such that

$$J_{\mathrm{X}}(Z)Z'^2 + \tfrac{1}{2}\{Z, z\} = \frac{1}{2}\frac{g}{H} - \frac{1}{4}\frac{H''}{H} + \frac{3}{16}\frac{H'^2}{H^2} \tag{58}$$

for all I. Z may, of course, and in fact will, depend upon I.

First we analyse the structure of movable poles on the left- and right-hand sides of (58).

The movable (and therefore simple) zeros of $P + IQ$ give rise to movable poles of the second order on the right-hand side of (58) with coefficients equal to $\frac{3}{16}$. Poles on the left-hand side of (58) are also necessarily of second order and arise from zeros of Z, $Z-1$ and Z' or form poles of Z. The coefficients of poles arising from zeros and poles of Z and $Z-1$, of order α, have the form $\frac{1}{4}(1-\upsilon^{-2}\alpha^2)$ for $\alpha \in \mathbb{N}$ and $\upsilon = \upsilon_i$ some $i = 1, 2$ or 3, whilst those arising from zeros of Z', of order τ, have the form $-(\frac{1}{2}\tau + \frac{1}{4}\tau^2)$ for $\tau \in \mathbb{N}$. These statements are easily verified by direct calculation [11]. Now the equation

$$-\frac{\tau}{2} - \frac{\tau^2}{4} = \frac{3}{16} \tag{59}$$

has no solution in $\mathbb{N}$. Therefore Z' *has no movable zeros*. On the other hand the equation

$$\frac{1}{4}\left(1 - \frac{\alpha^2}{\upsilon^2}\right) = \frac{3}{16} \tag{60}$$

has solution $\alpha = \upsilon/2 \in \mathbb{N}$ only if υ is even. Therefore *movable zeros of Z, $Z-1$ and movable poles of Z can only occur for even values of υ_2, υ_1 and υ_3 respectively. Moreover the degree of the zero or pole in Z is given by $\frac{1}{2}\upsilon_i$.*

We proceed to study the five cases in order of complexity.

5.1. Cases III, V

Since these cases are similar in having only one even exponent ($\upsilon_1 = 2$) we may treat them together. All movable zeros of $P + IQ$ are zeros of $Z - 1$ so that $Z = 1 + (P + IQ)\tilde{Z}$, where $\tilde{Z}$ has fixed poles and zeros. This Z must have only fixed zeros and poles, since υ_2 and υ_3 are both odd, and this is sufficient for Z' to have only fixed zeros.

We may write $Z = \eta(I)A(z)$ and $\tilde{Z} = \xi(I)B(z)$ where A and B are rational functions of z alone and η and ξ functions of I alone. Then η and ξ are related by

$$\xi^{-1}\eta A - \xi^{-1} - PB - IQB = 0. \tag{61}$$

Since A, P and Q are not constant functions of z both ξ^{-1} and $\xi^{-1}\eta$ must be linear functions of I with constant coefficients. Making such substitutions we obtain for Z a two-parameter family of solutions, homographic functions of I and $w = P/Q$,

$$Z_{k,l}(w; I) = \frac{k+I}{l+I}\,\frac{w-l}{w-k} \tag{62}$$

where $k \neq l$ and $k, l \neq -I$. If we make this substitution in (58), using the transformation law (45) and noting that $\{Z, w\} = 0$, we obtain

$$\begin{aligned}
\frac{1}{4}\bigg[&\left(1-\frac{1}{\upsilon_2^2}\right)\frac{1}{(w-l)^2}+\left(1-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-k)^2}+\left(1-\frac{1}{\upsilon_1^2}\right)\frac{1}{(w+I)^2}\\
&-\left(1+\frac{1}{\upsilon_1^2}-\frac{1}{\upsilon_3^2}-\frac{1}{\upsilon_2^2}\right)\frac{1}{(w-k)(w-l)}\\
&-\left(1+\frac{1}{\upsilon_2^2}-\frac{1}{\upsilon_1^2}-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-k)(w+I)}\\
&-\left(1+\frac{1}{\upsilon_3^2}-\frac{1}{\upsilon_2^2}-\frac{1}{\upsilon_1^2}\right)\frac{1}{(w-l)(w+I)}\bigg]w'^2+\tfrac{1}{2}\{w, z\}\\
&=\frac{g}{w+I}-\frac{1}{4}\frac{w''}{w+I}+\frac{3}{16}\frac{w'^2}{(w+I)^2}.
\end{aligned} \tag{63}$$

Note that this equation is no longer in exactly the canonical form (57) because the homographic transformation (62) has moved the singular point at infinity into the finite part of the complex line. Since $\upsilon_1 = 2$ the second-order pole at $w = -I$ cancels between the two sides as expected. Multiplying by $w + I$ gives an equation linear in I and, because g must be independent of I, we may equate coefficients to zero to get an equation for w,

$$\begin{aligned}
\{w, z\} = -\frac{w'^2}{2}\bigg[&\left(1-\frac{1}{\upsilon_2^2}\right)\frac{1}{(w-l)^2}+\left(1-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-k)^2}\\
&-\left(1+\frac{1}{\upsilon_1^2}-\frac{1}{\upsilon_2^2}-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-l)(w-k)}\bigg]
\end{aligned} \tag{64}$$

and an equation for g,

$$g = \frac{1}{4}w'' - \frac{w'^2}{4}\left[\left(1+\frac{1}{\upsilon_2^2}-\frac{1}{\upsilon_1^2}-\frac{1}{\upsilon_3^2}\right)\frac{1}{w-k}+\left(1+\frac{1}{\upsilon_3^2}-\frac{1}{\upsilon_1^2}-\frac{1}{\upsilon_2^2}\right)\frac{1}{w-l}\right]. \tag{65}$$

To solve (64), interchange dependent and independent variable to obtain

$$\{z, w\} = \frac{1}{2}\left[\left(1-\frac{1}{\upsilon_2^2}\right)\frac{1}{(w-l)^2}+\left(1-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-k)^2} - \left(1+\frac{1}{\upsilon_1^2}-\frac{1}{\upsilon_2^2}-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-l)(w-k)}\right] \tag{66}$$

$z(w)$ is then the ratio of solutions to the second-order linear equation

$$\frac{d^2\zeta}{dw^2}+\frac{1}{4}\left[\left(1-\frac{1}{\upsilon_2^2}\right)\frac{1}{(w-l)^2}+\left(1-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-k)^2} - \left(1+\frac{1}{\upsilon_1^2}-\frac{1}{\upsilon_2^2}-\frac{1}{\upsilon_3^2}\right)\frac{1}{(w-l)(w-k)}\right]\zeta = 0 \tag{67}$$

which is precisely of algebraic type with a singularity at infinity in w. The homographic transformation which takes $w \to \hat{w}$ such that $l \to 0$, $k \to \infty$ and $\infty \to 1$,

$$\hat{w} = \frac{w-l}{w-k} = \frac{l+I}{k+I} Z_{k,l}(w; I) \tag{68}$$

reorders the poles into the standard form (57).

There are limits of (62) which should be noted because they give rise to coalescence of singularities which are assumed distinct in (64) and (65). Thus $Z_{\infty,0}(w; I) = -w/I$ and (67) becomes a simple Euler equation,

$$\frac{d^2\zeta}{dw^2}+\frac{1}{4}\left(1-\frac{1}{\upsilon_2^2}\right)\frac{1}{w^2}\zeta = 0 \tag{69}$$

with corresponding solution

$$w = \left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)^{\upsilon_2}. \tag{70}$$

Similarly the choice $Z_{0,\infty}(w; I)$ gives the same equations but with υ_3 replacing υ_2. These are the type of the examples given in [10].

Returning to the general case, (64) is solved in terms of the relevant polyhedral functions [11] for w, i.e. P/Q, as a function of z, from which $g(z)$ is found via (65) and hence $f(z)$ from (7).

5.2. Case IV

The movable zeros of $P+IQ$ are zeros of $Z-1$ of order one or poles of $Z-1$ of order two but never zeros of Z. Thus we consider a polynomial factorization of $P+IQ$ into factors F and G. If either of F or G has fixed zeros only then it must be constant, but coprimality of P and Q. This gives the possibilities $F=1$, $G=P+IQ$ and $G=1$, $F=P+IQ$. Assume then that all zeros of F and G are movable. Then $Z-1=G^{-2}F\tilde{Z}$, where $\tilde{Z}$ is a rational function the positions of whose zeros and poles are I independent. We require that the zeros of Z and Z' be fixed. Of course the poles of either may be movable.

Under this requirement there must exist rational functions A and B of z only, and functions η and ξ and I only, such that

$$G^2+F\tilde{Z} = \eta A \qquad (F\tilde{Z})'G-2(F\tilde{Z})G' = \xi B \tag{71}$$

the prime denoting a z-derivative, from which it follows that

$$A'G-2AG'=\eta^{-1}\xi B \tag{72}$$

$$(A^{-1/2}G)'=-\tfrac{1}{2}\eta^{-1}\xi A^{-3/2}B \tag{73}$$

and G is thus of the form

$$G=-\frac{1}{2}\frac{\zeta}{\eta}(H+\lambda)A^{1/2} \tag{74}$$

where $B=H'A^{3/2}$ and λ is a non-trivial function of I only, in order that G has movable zeros. G is a polynomial and therefore $A=C^2$ where C is a polynomial in z only. Also G has no fixed zeros and so $HA^{1/2}=D$ is a polynomial in z only coprime to C. Thus

$$\begin{aligned}G&=-\frac{1}{2}\frac{\zeta}{\eta}(D+\lambda C)\\ F\tilde{Z}&=\frac{1}{4}\frac{\zeta^2}{\eta^3}\left[D+\left(\lambda+\frac{2\eta^{3/2}}{\zeta}\right)C\right]\left[D+\left(\lambda-\frac{2\eta^{3/2}}{\zeta}\right)C\right].\end{aligned} \tag{75}$$

These functions satisfy the requirements on Z and Z'. We have now to make them satisfy the factorization $P+IQ=FG$.

Firstly, unless $\lambda=\pm 2\zeta^{-1}\eta^{3/2}$, the expression for $F\tilde{Z}$ has no fixed zeros and $\tilde{Z}$ must be 1. In either of these exceptional cases, $\tilde{Z}=D$, and the product FG has three terms linearly independent in I which cannot match $P+IQ$. Thus we have

$$P+IQ=\frac{\zeta^3}{8\eta^4}\left[D^3+3\lambda D^2C+\left(3\lambda^2-\frac{4\eta^3}{\zeta^2}\right)DC^2+\lambda\left(\lambda^2-\frac{4\eta^3}{\zeta^2}\right)C^3\right]. \tag{76}$$

If the coefficients are not all linear functions of I then the polynomials D^3, D^2C, etc., must be linearly dependent but this contradicts the coprimality of D and C. But it is not difficult to show that the coefficients cannot all be linear in I given that λ must have some I dependence.

This leaves us with only the possibilities $F=1$, $G=P+IQ$ and $F=P+IQ$, $G=1$. In the former case one cannot satisfy the first equation of (71) and the latter case is just that which obtained for cases III and V above. The same formulae then apply but with the new values of the exponents.

5.3. Case I

If N is odd there are no examples.

If $N=2n$ we must make a factorization FG of $P+IQ$ and write $Z-1=(F/G)^n\tilde{Z}$. Now Z and Z' must have fixed zeros and this can only be achieved for $n=1$. A similar argument to the previous case leads us to $F=1$ and $G=P+IQ$ or $G=1$ and $F=P+IQ$ or to $G=-\eta^{-1}\zeta(D+\mu A)$, $F=\eta^{-1}\zeta(D+\lambda A)$ with $\lambda-\mu=\zeta^{-1}\eta^2$ where, as before, D and A are coprime polynomials in z only and greek letters are functions of I only. It remains to fulfil the factorization condition

$$P+IQ=-\frac{\zeta^2}{\eta^2}(D+\mu A)(D+\lambda A). \tag{77}$$

This time, for certain forms of P and Q, the solution set is non-empty. For either $\mu+\lambda$ and $\mu\lambda$ are rational linear functions of I or the polynomials D^2, DC and C^2 are linearly independent over $\mathbb{C}$ which contradicts coprimality. μ and λ are then the zeros of a quadratic form whose coefficients are linear in I,

$$(a_0+b_0I)\xi^2+2(a_1+b_1I)\xi+(a_2+b_2I) \tag{78}$$

provided P and Q are quadratic forms in the arbitrary polynomials D and A,

$$P=-a_0D^2+2a_1DA-a_2A^2 \qquad Q=-b_0D^2+2b_1DA-b_2A^2. \tag{79}$$

A simple example arises if $P=P_1^2$ and $Q=-Q_1^2$, i.e. $F=P_1+\sqrt{I}\,Q_1$, $G=P_1-\sqrt{I}\,Q_1$.

Thus we arrive at three cases:

(i) $Z=1+(P+IQ)\tilde{Z}$, in which case we can return to the path followed in cases III and V.

(ii) $Z=1+(P+IQ)^{-1}\tilde{Z}$, and the usual type of calculation leads to $Z=(k+I)(w-l)(k-l)^{-1}(w+I)^{-1}$ where k and l may take the value infinity. Solving (58) in the same way as before,

$$\{w, z\}+\frac{3}{8}\frac{w'^2}{(w-k)^2}=0 \qquad g=\frac{1}{2}w''-\frac{3}{4}\frac{w'^2}{w-k} \tag{80}$$

which are easily solved for w and g.

(iii) $Z=(\mu-\lambda)(v+k)(k-\lambda)^{-1}(v+\mu)^{-1}$ where $v=D/A$ and k is a parameter which may take the value infinity. Solving (58) leads us to the equation for v,

$$\{v, z\}=\frac{3}{4}\left(\frac{Q_{vv}}{Q}-\frac{1}{2}\frac{Q_v^2}{Q^2}\right)v'^2 \tag{81}$$

where we have written Q as $-b_0v^2+2b_1v-b_2$. This equation, treated exactly as (64) was, leads us to a second-order equation with invariant $J_I(N=2)$, singularities at the zeros of Q in the v-plane (none at ∞), solvable via cyclic polynomials. Given this $v(z)$ and writing $w(v)=P/Q$, a ratio of quadratic functions, we find for g,

$$g=-\tfrac{1}{4}w_{vv}v'^2+\tfrac{1}{2}w_vv''-\frac{3}{4}\left(\frac{1}{v-v_0}+\frac{1}{v-v_1}\right)\,w_vv'^2 \tag{82}$$

v_0 and v_1 being the zeros of $Q(v)$.

5.4. *Case II*

Here the subcases N even and N odd must be treated quite distinctly.

Firstly suppose that N is odd. Then the movable zeros are zeros of Z and $Z-1$ the poles being fixed. So $Z=F\eta A$ and $Z=1+G\zeta B$ where FG is a factorization of $P+IQ$ and A and B are rational functions of z alone. Then the usual argument, in addition to the cases $F=P+IQ$, $G=1$ and $G=P+IQ$, $F=1$ as before, yields

$$Z=\frac{u+\lambda}{\lambda-\zeta} \tag{83}$$

where P and Q are quadratic in the polynomial $u(z)$

$$P=p_0u^2-2p_1u+p_2 \qquad Q=q_0u^2-2q_1u+q_2 \tag{84}$$

and λ and ζ are the zeros (in the u-plane) of $P+IQ$. As a function of z, u must satisfy

$$\{u, z\}=\left(\frac{1}{2}+\frac{1}{4N^2}\right)\frac{Q_{uu}}{Q}u'^2+\frac{3}{8}\frac{Q_u}{Q^2}u'^2 \tag{85}$$

which is equivalent to a second-order equation having singularities at the roots of Q and at infinity and solvable in terms of dihedral polynomials. So, given $u(z)$ and writing $w = P/Q$ as normal,

$$g = \frac{1}{4} w_u u'' - \frac{1}{4N^2} w_{uu} u'^2 - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{N^2}\right) \frac{Q_u}{Q} w_u u'^2. \tag{86}$$

This brings us finally to the case when N is even: $N = 2n$. Here the zeros of $P + IQ$ can be distributed between the zeros *and* poles of Z and $Z - 1$. Suppose we have a factorization $P + IQ = FGH$, then we will have

$$Z = \eta A H^{-n} F = 1 + \zeta B H^{-n} G. \tag{87}$$

As before we may have one of F, G and H equal to unity, with $n = 1$. Thus if $H = 1$, Z will have the form (83), if $F = 1$, it will have the form given under the cyclic case and, if $G = 1$, $Z - 1$ will have this form. We may also have $F = P + IQ$, $G = 1$, $H = 1$ and so on. These all lead to similar formulae to those obtained already with the same forms for P and Q but new values of the exponents.

The new possibility which arises in this case is that P and Q are homogeneous cubic functions of arbitrary polynomials in z,

$$P = p_0 D^3 - 3p_1 D^2 A + 3p_2 D A^2 - p_3 A^3 \qquad Q = q_0 D^3 - 3q_1 D^2 A + 3q_2 D A^2 - q_3 A^3 \tag{88}$$

with $n = 1$ and Z given by

$$Z = \frac{(s + \lambda_1)(\lambda_3 - \lambda_2)}{(s + \lambda_3)(\lambda_1 - \lambda_2)} \tag{89}$$

where $s = D/A$ and the λ_i are the roots of the cubic equation

$$(p_0 + Iq_0)\lambda^3 + 3(p_1 + Iq_1)\lambda^2 + 3(p_2 + Iq_2)\lambda + (p_3 + Iq_3) = 0. \tag{90}$$

This *appears* to be the most general solution to the criteria satisfied by Z, but to show that this is so is not as easy as in the previous cases.

Then we find that

$$\{z, s\} = \frac{3}{16}\left(3\frac{Q_{ss}}{Q} - 2\frac{Q_s^2}{Q^2}\right) \tag{91}$$

where $Q = q_0 s^3 - 3q_1 s^2 + 3q_2 s - q_3$ has singularities at the roots of Q in the s-plane and is solvable in terms of dihedral polynomials. Putting $P = wQ$, as usual we obtain for g the expression

$$g = \tfrac{1}{4} w_s s'' - \tfrac{1}{32} s'^2 \left(w_{ss} + 2w\frac{Q_s^2}{Q^2} - 6w_s \frac{Q_s}{Q} \right). \tag{92}$$

5.5. Summary

In summary, then, we conclude that in each of the cases I–V one has a number of subcases corresponding to the number of even exponents. P and Q can be polynomial functions, of degree less than or equal to this number, in other polynomials which are themselves constructed from solutions to linear, second-order ODEs (independent of I) belonging to the relevant class (I–V). The rational function Z is in each case a homographic transformation in these polynomials. The general integral of such a Ermakov system will be algebraic in solutions of the linear time-dependent oscillator (2).

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